A Simple and Intuitive Coverage of The Fundamental Theorems of Asset Pricing

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Overview

Simple Setting for Intuitive Understanding

- Single-period setting (two time points $t = 0$ and $t = 1$)
- $t = 0$ has a single state (we'll call it "Spot" state)
- $t = 1$ has n random states represented by $\Omega = \{\omega_1, \ldots, \omega_n\}$
- With probability distribution $\mu : \Omega \rightarrow [0, 1]$, i.e, $\sum_{i=1}^{n} \mu(\omega_i) = 1$
- $m + 1$ fundamental assets $A_0, A_1, \ldots, A_m$
- Spot Price (at $t = 0$) of A_j denoted $S_j^{(0)}$ for all $j = 0, 1, \ldots, m$
- Price of A_j in state ω_i denoted $S_j^{(i)}$ for all $j = 0, \ldots, m, i = 1, \ldots, n$
- All asset prices are assumed to be real numbers, i.e. in $\mathbb{R}$
- A_0 is a special asset known as risk-free asset with $S_0^{(0)}$ normalized to 1
- $S_0^{(i)} = e^r$ for all $i = 1, \ldots, n$ where r is the constant risk-free rate
- e^{-r} is the risk-free discount factor to represent "time value of money"

Portfolios

- A portfolio is a vector $\theta = (\theta_0, \theta_1, \ldots, \theta_m) \in \mathbb{R}^{m+1}$
- θ_j is the number of units held in asset A_j for all $j = 0, 1, \ldots, m$
- Spot Value (at $t = 0$) of portfolio θ denoted $V_\theta^{(0)}$ is:

$$V_\theta^{(0)} = \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)}$$

- Value of portfolio θ in state ω_i (at $t = 1$) denoted $V_\theta^{(i)}$ is:

$$V_\theta^{(i)} = \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} \text{ for all } i = 1, \ldots, n$$

Arbitrage Portfolio

- An Arbitrage Portfolio θ is one that "makes money from nothing"
- Formally, a portfolio θ such that:
  - $V_\theta^{(0)} \leq 0$
  - $V_\theta^{(i)} \geq 0$ for all $i = 1, \ldots, n$
  - $\exists i$ in $1, \ldots, n$ such that $\mu(\omega_i) > 0$ and $V_\theta^{(i)} > 0$
- So we never end with less value than what we start with and we end with expected value greater than what we start with
- Arbitrage allows market participants to make infinite returns
- In an efficient market, arbitrage disappears as participants exploit it
- Hence, Finance Theory typically assumes "arbitrage-free" markets

Risk-Neutral Probability Measure

- Consider a Probability Distribution $\pi : \Omega \rightarrow [0,1]$ such that

$$\pi(\omega_i) = 0 \text{ if and only if } \mu(\omega_i) = 0 \text{ for all } i = 1, \ldots, n$$

- Then, π is a Risk-Neutral Probability Measure if:

$$S_j^{(0)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot S_j^{(i)} \text{ for all } j = 0, 1, \ldots, m \tag{1}$$

- So for each of the $m+1$ assets, the asset spot price (at $t = 0$) is the discounted expectation (under π) of the asset price at $t = 1$
- π is an artificial construct to connect expectation of asset prices at $t = 1$ to their spot prices by the risk-free discount factor e^{-r}

1st Fundamental Theorem of Asset Pricing (1st FTAP)

Theorem

1st FTAP: Our simple setting will not admit arbitrage portfolios if and only if there exists a Risk-Neutral Probability Measure.

- First we prove the easy implication:
  Existence of Risk-Neutral Measure $\Rightarrow$ Arbitrage-free
- Assume there is a risk-neutral measure π
- Then, for each portfolio $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$,

$$V_\theta^{(0)} = \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} = \sum_{j=0}^{m} \theta_j \cdot e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot S_j^{(i)}$$

$$= e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot V_\theta^{(i)}$$

1st Fundamental Theorem of Asset Pricing (1st FTAP)

- So the portfolio spot value is the discounted expectation (under π) of the portfolio value at $t = 1$
- For any portfolio θ, if the following two conditions are satisfied:
  - $V_\theta^{(i)} \geq 0$ for all $i = 1, \ldots, n$
  - $\exists i$ in $1, \ldots, n$ such that $\mu(\omega_i) > 0 (\Rightarrow \pi(\omega_i) > 0)$ and $V_\theta^{(i)} > 0$

  Then,

$$V_\theta^{(0)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot V_\theta^{(i)} > 0$$

- This eliminates the the possibility of arbitrage for any portfolio θ
- The other implication (Arbitrage-free $\Rightarrow$ Existence of Risk-Neutral Measure) is harder to prove and covered in Appendix 1

Derivatives, Replicating Portfolios and Hedges

- A Derivative D (in this simple setting) is a vector payoff at $t = 1$:

$$(V_D^{(1)}, V_D^{(2)}, \ldots, V_D^{(n)})$$

where $V_D^{(i)}$ is the payoff of the derivative in state ω_i for all $i = 1, \ldots, n$

- Portfolio $\theta \in \mathbb{R}^{m+1}$ is a *Replicating Portfolio* for derivative D if:

$$V_D^{(i)} = \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} \text{ for all } i = 1, \ldots, n \qquad (2)$$

- The negatives of the components $(\theta_0, \theta_1, \ldots, \theta_m)$ are known as the *hedges* for D since they can be used to offset the risk in the payoff of D at $t = 1$

2nd Fundamental Theorem of Asset Pricing (2nd FTAP)

An arbitrage-free market is said to be *Complete* if every derivative in the market has a replicating portfolio.

Theorem

2nd FTAP: A market is Complete in our simple setting if and only if there is a unique risk-neutral probability measure.

Proof in Appendix 2. Together, the FTAPs classify markets into:

1. Complete (arbitrage-free) market $\Leftrightarrow$ Unique risk-neutral measure
2. Market with arbitrage $\Leftrightarrow$ No risk-neutral measure
3. Incomplete (arbitrage-free) market $\Leftrightarrow$ Multiple risk-neutral measures

The next topic is derivatives pricing that is based on the concepts of *replication of derivatives* and *risk-neutral measures*, and so is tied to the concepts of *arbitrage* and *completeness*.

Positions involving a Derivative

- Before getting into Derivatives Pricing, we need to define a *Position*
- We define a *Position* involving a derivative D as the combination of holding some units in D and some units in $A_0, A_1, \ldots, A_m$
- *Position* is an extension of the Portfolio concept including a derivative
- Formally denoted as $\gamma_D = (\alpha, \theta_0, \theta_1, \ldots, \theta_m) \in \mathbb{R}^{m+2}$
- α is the units held in derivative D
- θ_j is the units held in A_j for all $j = 0, 1, \ldots, m$
- Extend the definition of Portfolio Value to Position Value
- Extend the definition of Arbitrage Portfolio to Arbitrage Position

Derivatives Pricing: Elimination of candidate prices

- We will consider candidate prices (at $t = 0$) for a derivative D
- Let $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$ be a replicating portfolio for D
- Consider the candidate price $\sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} - x$ for D for any $x > 0$
- Position $(1, -\theta_0 + x, -\theta_1, \ldots, -\theta_m)$ has value $x \cdot e^r > 0$ in each of the states at $t = 1$
- But this position has spot ($t = 0$) value of 0, which means this is an Arbitrage Position, rendering this candidate price invalid
- Consider the candidate price $\sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} + x$ for D for any $x > 0$
- Position $(-1, \theta_0 + x, \theta_1, \ldots, \theta_m)$ has value $x \cdot e^r > 0$ in each of the states at $t = 1$
- But this position has spot ($t = 0$) value of 0, which means this is an Arbitrage Position, rendering this candidate price invalid
- So every candidate price for D other than $\sum_{j=0}^{m} \theta_j \cdot S_j^{(0)}$ is invalid

Derivatives Pricing: Remaining candidate price

- Having eliminated various candidate prices for D, we now aim to *establish* the remaining candidate price:

$$V_D^{(0)} = \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} \tag{3}$$

  where $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$ is a replicating portfolio for D
- To eliminate prices, our only assumption was that D can be replicated
- This can happen in a complete market or in an arbitrage market
- To establish remaining candidate price $V_D^{(0)}$, we need to assume market is complete, i.e., there is a unique risk-neutral measure π
- Candidate price $V_D^{(0)}$ can be expressed as the discounted expectation (under π) of the payoff of D at $t = 1$, i.e.,

$$V_D^{(0)} = \sum_{j=0}^{m} \theta_j \cdot e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot S_j^{(i)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot V_D^{(i)} \tag{4}$$

Derivatives Pricing: Establishing remaining candidate price

- Now consider an *arbitrary portfolio* $\beta = (\beta_0, \beta_1, \ldots, \beta_m)$
- Define a position $\gamma_D = (\alpha, \beta_0, \beta_1, \ldots, \beta_m)$
- Spot Value (at $t = 0$) of position γ_D denoted $V_{\gamma_D}^{(0)}$ is:

$$V_{\gamma_D}^{(0)} = \alpha \cdot V_D^{(0)} + \sum_{j=0}^{m} \beta_j \cdot S_j^{(0)} \tag{5}$$

  where $V_D^{(0)}$ is the remaining candidate price
- Value of position γ_D in state ω_i (at $t = 1$), denoted $V_{\gamma_D}^{(i)}$, is:

$$V_{\gamma_D}^{(i)} = \alpha \cdot V_D^{(i)} + \sum_{j=0}^{m} \beta_j \cdot S_j^{(i)} \text{ for all } i = 1, \ldots, n \tag{6}$$

- Combining the linearity in equations (1), (4), (5), (6), we get:

$$V_{\gamma_D}^{(0)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot V_{\gamma_D}^{(i)} \tag{7}$$

Derivatives Pricing: Establishing remaining candidate price

- So the position spot value is the discounted expectation (under π) of the position value at $t = 1$
- For any γ_D (containing any arbitrary portfolio β) and with $V_D^{(0)}$ as the candidate price for D, if the following two conditions are satisfied:
  - $V_{\gamma_D}^{(i)} \geq 0$ for all $i = 1, \ldots, n$
  - $\exists i$ in $1, \ldots, n$ such that $\mu(\omega_i) > 0 (\Rightarrow \pi(\omega_i) > 0)$ and $V_{\gamma_D}^{(i)} > 0$

  Then,
  $$V_{\gamma_D}^{(0)} = e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot V_{\gamma_D}^{(i)} > 0$$
- This eliminates arbitrage possibility for remaining candidate price $V_D^{(0)}$
- So we have eliminated all prices other than $V_D^{(0)}$, and we have established the price $V_D^{(0)}$, proving that it should be *the* price of D
- The above arguments assumed a complete market, but what about an incomplete market or a market with arbitrage?

Incomplete Market (Multiple Risk-Neutral Measures)

- Recall: Incomplete market means some derivatives can't be replicated
- Absence of replicating portfolio precludes usual arbitrage arguments
- 2nd FTAP says there are multiple risk-neutral measures
- So, multiple derivative prices (each consistent with no-arbitrage)
- *Superhedging* (outline in Appendix 3) provides bounds for the prices
- But often these bounds are not tight and so, not useful in practice
- The alternative approach is to identify hedges that maximize Expected Utility of the derivative together with the hedges
- For an appropriately chosen market/trader Utility function
- Utility function is a specification of reward-versus-risk preference that effectively chooses the risk-neutral measure and (hence, Price)
- We outline the Expected Utility approach in Appendix 4

Multiple Replicating Portfolios (Arbitrage Market)

- Assume there are replicating portfolios α and β for D with

$$\sum_{j=0}^{m} \alpha_j \cdot S_j^{(0)} - \sum_{j=0}^{m} \beta_j \cdot S_j^{(0)} = x > 0$$

- Consider portfolio $\theta = (\beta_0 - \alpha_0 + x, \beta_1 - \alpha_1, \ldots, \beta_m - \alpha_m)$

$$V_\theta^{(0)} = \sum_{j=0}^{m} (\beta_j - \alpha_j) \cdot S_j^{(0)} + x \cdot S_0^{(0)} = -x + x = 0$$

$$V_\theta^{(i)} = \sum_{j=0}^{m} (\beta_j - \alpha_j) \cdot S_j^{(i)} + x \cdot S_0^{(i)} = x \cdot e^r > 0 \text{ for all } i = 1, \ldots, n$$

- So θ is an Arbitrage Portfolio $\Rightarrow$ market with no risk-neutral measure
- Also note from previous elimination argument that every candidate price other than $\sum_{j=0}^{m} \alpha_j \cdot S_j^{(0)}$ is invalid and every candidate price other than $\sum_{j=0}^{m} \beta_j \cdot S_j^{(0)}$ is invalid, so D has no valid price at all

Market with 2 states and 1 Risky Asset

- Consider a market with $m = 1$ and $n = 2$
- Assume $S_1^{(1)} < S_1^{(2)}$
- No-arbitrage requires $S_1^{(1)} \leq S_1^{(0)} \cdot e^r \leq S_1^{(2)}$
- Assuming absence of arbitrage and invoking 1st FTAP, there exists a risk-neutral probability measure π such that:

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)})$$

$$\pi(\omega_1) + \pi(\omega_2) = 1$$

- This implies:

$$\pi(\omega_1) = \frac{S_1^{(2)} - S_1^{(0)} \cdot e^r}{S_1^{(2)} - S_1^{(1)}}$$

$$\pi(\omega_2) = \frac{S_1^{(0)} \cdot e^r - S_1^{(1)}}{S_1^{(2)} - S_1^{(1)}}$$

Market with 2 states and 1 Risky Asset (continued)

- We can use these probabilities to price a derivative D as:

$$V_D^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot V_D^{(1)} + \pi(\omega_2) \cdot V_D^{(2)})$$

- Now let us try to form a replicating portfolio (θ_0, θ_1) for D

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)}$$
$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)}$$

- Solving this yields Replicating Portfolio (θ_0, θ_1) as follows:

$$\theta_0 = e^{-r} \cdot \frac{V_D^{(1)} \cdot S_1^{(2)} - V_D^{(2)} \cdot S_1^{(1)}}{S_1^{(2)} - S_1^{(1)}} \text{ and } \theta_1 = \frac{V_D^{(2)} - V_D^{(1)}}{S_1^{(2)} - S_1^{(1)}}$$

- This means this is a Complete Market
- Note that the derivative price can also be expressed as:

$$V_D^{(0)} = \theta_0 + \theta_1 \cdot S_1^{(0)}$$

Market with 3 states and 1 Risky Asset

- Consider a market with $m = 1$ and $n = 3$
- Assume $S_1^{(1)} < S_1^{(2)} < S_1^{(3)}$
- No-arbitrage requires $S_1^{(1)} \leq S_1^{(0)} \cdot e^r \leq S_1^{(3)}$
- Assuming absence of arbitrage and invoking 1st FTAP, there exists a risk-neutral probability measure π such that:

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)} + \pi(\omega_3) \cdot S_1^{(3)})$$

$$\pi(\omega_1) + \pi(\omega_2) + \pi(\omega_3) = 1$$

- 2 equations & 3 variables $\Rightarrow$ multiple solutions for π
- Each of these solutions for π provides a valid price for a derivative D

$$V_D^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot V_D^{(1)} + \pi(\omega_2) \cdot V_D^{(2)} + \pi(\omega_3) \cdot V_D^{(3)})$$

Market with 3 states and 1 Risky Asset (continued)

- Now let us try to form a replicating portfolio (θ_0, θ_1) for D

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)}$$

$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)}$$

$$V_D^{(3)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(3)}$$

- 3 equations & 2 variables $\Rightarrow$ no replication for *some* D
- This means this is an Incomplete Market
- Don't forget that we have multiple risk-neutral probability measures
- Meaning we have multiple valid prices for derivatives

Market with 2 states and 2 Risky Assets

- Consider a market with $m = 2$ and $n = 3$
- Assume $S_1^{(1)} < S_1^{(2)}$ and $S_2^{(1)} < S_2^{(2)}$
- Let us try to determine a risk-neutral probability measure π:

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)})$$

$$S_2^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_2^{(1)} + \pi(\omega_2) \cdot S_2^{(2)})$$

$$\pi(\omega_1) + \pi(\omega_2) = 1$$

- 3 equations & 2 variables $\Rightarrow$ no risk-neutral measure π
- Let's try to form a replicating portfolio $(\theta_0, \theta_1, \theta_2)$ for a derivative D

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)} + \theta_2 \cdot S_2^{(1)}$$

$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)} + \theta_2 \cdot S_2^{(2)}$$

Market with 2 states and 2 Risky Assets (continued)

- 2 equations & 3 variables $\Rightarrow$ multiple replicating portfolios
- Each such replicating portfolio yields a price for D as:

$$V_D^{(0)} = \theta_0 + \theta_1 \cdot S_1^{(0)} + \theta_2 \cdot S_2^{(0)}$$

- Select two such replicating portfolios with different $V_D^{(0)}$
- Combination of these replicating portfolios is an Arbitrage Portfolio
  - They cancel off each other's price in each $t = 1$ states
  - They have a combined negative price at $t = 0$
- So this is a market that admits arbitrage (no risk-neutral measure)

Summary

3 cases:

1. *Complete market*
   - Unique replicating portfolio for derivatives
   - Unique risk-neutral measure, meaning we have unique derivatives prices
2. *Arbitrage-free but incomplete market*
   - Not all derivatives can be replicated
   - Multiple risk-neutral measures, meaning we can have multiple valid prices for derivatives
3. *Market with Arbitrage*
   - Derivatives have multiple replicating portfolios (that when combined causes arbitrage)
   - No risk-neutral measure, meaning derivatives cannot be priced

General Theory for Derivatives Pricing

- The theory for our simple setting extends nicely to the general setting
- Instead of $t = 0, 1$, we consider $t = 0, 1, \ldots, T$
- The model is a "recombining tree" of state transitions across time
- The idea of Arbitrage applies over multiple time periods
- Risk-neutral measure for each state at each time period
- Over multiple time periods, we need a *Dynamic Replicating Portfolio* to rebalance asset holdings ("self-financing trading strategy")
- We obtain prices and replicating portfolio at each time in each state
- By making time period smaller and smaller, the model turns into a stochastic process (in continuous time)
- Classical Financial Math theory based on stochastic calculus but has essentially the same ideas we developed for our simple setting

Appendix 1: Arbitrage-free $\Rightarrow \exists$ a risk-neutral measure

- We will prove that if a risk-neutral probability measure doesn't exist, there exists an arbitrage portfolio
- Let $\mathbb{V} \subset \mathbb{R}^m$ be the set of vectors $(s_1, \ldots, s_m)$ such that

$$s_j = e^{-r} \cdot \sum_{i=1}^{n} \mu(\omega_i) \cdot S_j^{(i)} \text{ for all } j = 1, \ldots, m$$

  spanning over all possible probability distributions $\mu : \Omega \rightarrow [0, 1]$
- $\mathbb{V}$ is a bounded, closed, convex polytope in $\mathbb{R}^m$
- If a risk-neutral measure doesn't exist, $(S_1^{(0)}, \ldots, S_m^{(0)}) \notin \mathbb{V}$
- Hyperplane Separation Theorem implies that there exists a non-zero vector $(\theta_1, \ldots, \theta_m)$ such that for any $v = (v_1, \ldots, v_m) \in \mathbb{V}$,

$$\sum_{j=1}^{m} \theta_j \cdot v_j > \sum_{j=1}^{m} \theta_j \cdot S_j^{(0)}$$

Appendix 1: Arbitrage-free $\Rightarrow \exists$ a risk-neutral measure

- In particular, consider vectors v corresponding to the corners of $\mathbb{V}$, those for which the full probability mass is on a particular $\omega_i \in \Omega$, i.e.,

$$\sum_{j=1}^{m} \theta_j \cdot (e^{-r} \cdot S_j^{(i)}) > \sum_{j=1}^{m} \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \ldots, n$$

- Choose a $\theta_0 \in \mathbb{R}$ such that:

$$\sum_{j=1}^{m} \theta_j \cdot (e^{-r} \cdot S_j^{(i)}) > -\theta_0 > \sum_{j=1}^{m} \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \ldots, n$$

- Therefore,

$$e^{-r} \cdot \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} > 0 > \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \ldots, n$$

- This means $(\theta_0, \theta_1, \ldots, \theta_m)$ is an arbitrage portfolio $\square$

Appendix 2: Proof of 2nd FTAP

- We will first prove that in an arbitrage-free market, if every derivative has a replicating portfolio, there is a unique risk-neutral measure π
- We define n special derivatives (known as *Arrow-Debreu securities*), one for each random state in Ω at $t = 1$
- We define the time $t = 1$ payoff of *Arrow-Debreu security* D_k (for each of $k = 1, \ldots, n$) in state ω_i as $\mathbb{I}_{i=k}$ for all $i = 1, \ldots, n$.
- Since each derivative has a replicating portfolio, let $\theta^{(k)} = (\theta_0^{(k)}, \theta_1^{(j)}, \ldots, \theta_m^{(k)})$ be the replicating portfolio for D_k.
- With usual no-arbitrage argument, the price (at $t = 0$) of D_k is

$$\sum_{j=0}^{m} \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \ldots, n$$

Appendix 2: Proof of 2nd FTAP

- Now let us try to solve for an unknown risk-neutral probability measure $\pi : \Omega \to [0,1]$, given the above prices for $D_k, k = 1, \ldots, n$

$$e^{-r} \cdot \sum_{i=1}^{n} \pi(\omega_i) \cdot \mathbb{I}_{i=k} = e^{-r} \cdot \pi(\omega_k) = \sum_{j=0}^{m} \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \ldots, n$$

$$\Rightarrow \pi(\omega_k) = e^{r} \cdot \sum_{j=0}^{m} \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \ldots, n$$

- This yields a unique solution for the risk-neutral probability measure π
- Next, we prove the other direction of the 2nd FTAP
- To prove: if there exists a risk-neutral measure π and if there exists a derivative D with no replicating portfolio, we can construct a risk-neutral measure different than π

Appendix 2: Proof of 2nd FTAP

- Consider the following vectors in the vector space $\mathbb{R}^n$

$$v = (V_D^{(1)}, \ldots, V_D^{(n)}) \text{ and } s_j = (S_j^{(1)}, \ldots, S_j^{(n)}) \text{ for all } j = 0, 1, \ldots, m$$

- Since D does not have a replicating portfolio, v is not in the span of $s_0, s_1, \ldots, s_m$, which means $s_0, s_1, \ldots, s_m$ do not span $\mathbb{R}^n$
- Hence $\exists$ a non-zero vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ orthogonal to each of $s_0, s_1, \ldots, s_m$, i.e.,

$$\sum_{i=1}^{n} u_i \cdot S_j^{(i)} = 0 \text{ for all } j = 0, 1, \ldots, n \tag{8}$$

- Note that $S_0^{(i)} = e^r$ for all $i = 1, \ldots, n$ and so,

$$\sum_{i=1}^{n} u_i = 0 \tag{9}$$

Appendix 2: Proof of 2nd FTAP

- Define $\pi' : \Omega \rightarrow \mathbb{R}$ as follows (for some $\epsilon > 0 \in \mathbb{R}$):

$$\pi'(\omega_i) = \pi(\omega_i) + \epsilon \cdot u_i \text{ for all } i = 1, \ldots, n \tag{10}$$

- To establish π' as a risk-neutral measure different than π, note:
  - Since $\sum_{i=1}^n \pi(\omega_i) = 1$ and since $\sum_{i=1}^n u_i = 0$, $\sum_{i=1}^n \pi'(\omega_i) = 1$
  - Construct $\pi'(\omega_i) > 0$ for each i where $\pi(\omega_i) > 0$ by making $\epsilon > 0$ sufficiently small, and set $\pi'(\omega_i) = 0$ for each i where $\pi(\omega_i) = 0$
  - From Eq (8) and Eq (10), we derive:

$$\sum_{i=1}^n \pi'(\omega_i) \cdot S_j^{(i)} = \sum_{i=1}^n \pi(\omega_i) \cdot S_j^{(i)} = e^r \cdot S_j^{(0)} \text{ for all } j = 0, 1, \ldots, m$$

$\square$

Appendix 3: Superhedging

- Superhedging is a technique to price in incomplete markets
- Where one cannot replicate & there are multiple risk-neutral measures
- The idea is to create a portfolio of fundamental assets whose Value *dominates* the derivative payoff in *all* states at $t = 1$
- Superhedge Price is the smallest possible Portfolio Spot ($t = 0$) Value among all such Derivative-Payoff-Dominating portfolios
- This is a constrained linear optimization problem:

$$\min_{\theta} \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} \text{ such that } \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} \geq V_D^{(i)} \text{ for all } i = 1, \ldots, n \quad (11)$$

- Let $\theta^* = (\theta_0^*, \theta_1^*, \ldots, \theta_m^*)$ be the solution to Equation (11)
- Let SP be the Superhedge Price $\sum_{j=0}^{m} \theta_j^* \cdot S_j^{(0)}$

Appendix 3: Superhedging

- Establish feasibility and define Lagrangian $J(\theta, \lambda)$

$$J(\theta, \lambda) = \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} + \sum_{i=1}^{n} \lambda_i \cdot (V_D^{(i)} - \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)})$$

- So there exists $\lambda = (\lambda_1, \ldots, \lambda_n)$ that satisfy these KKT conditions:

$$\lambda_i \geq 0 \text{ for all } i = 1, \ldots, n$$

$$\lambda_i \cdot (V_D^{(i)} - \sum_{j=0}^{m} \theta_j^* \cdot S_j^{(i)}) \text{ for all } i = 1, \ldots, n \text{ (Complementary Slackness)}$$

$$\nabla_\theta J(\theta^*, \lambda) = 0 \Rightarrow S_j^{(0)} = \sum_{i=1}^{n} \lambda_i \cdot S_j^{(i)} \text{ for all } j = 0, 1, \ldots, m$$

Appendix 3: Superhedging

- This implies $\lambda_i = e^{-r} \cdot \pi(\omega_i)$ for all $i = 1, \ldots, n$ for a risk-neutral probability measure $\pi : \Omega \to [0, 1]$ (λ is "discounted probabilities")
- Define Lagrangian Dual $L(\lambda) = \inf_\theta J(\theta, \lambda)$. Then, Superhedge Price

$$SP = \sum_{j=0}^{m} \theta_j^* \cdot S_j^{(0)} = \sup_\lambda L(\lambda) = \sup_\lambda \inf_\theta J(\theta, \lambda)$$

- Complementary Slackness and some linear algebra over the space of risk-neutral measures $\pi : \Omega \to [0, 1]$ enables us to argue that:

$$SP = \sup_\pi \sum_{i=1}^{n} \pi(\omega_i) \cdot V_D^{(i)}$$

Appendix 3: Superhedging

- Likewise, the *Subhedging* price SB is defined as:

$$\max_{\theta} \sum_{j=0}^{m} \theta_j \cdot S_j^{(0)} \text{ such that } \sum_{j=0}^{m} \theta_j \cdot S_j^{(i)} \leq V_D^{(i)} \text{ for all } i = 1, \ldots, n \quad (12)$$

- Likewise arguments enable us to establish:

$$SB = \inf_{\pi} \sum_{i=1}^{n} \pi(\omega_i) \cdot V_D^{(i)}$$

- This gives a lower bound of SB and an upper bound of SP, meaning:
    - A price outside these bounds leads to an arbitrage
    - Valid prices must be established within these bounds

Appendix 4: Maximization of Expected Utility

- *Maximization of Expected Utility* is a technique to establish pricing and hedging in incomplete markets
- Based on a concave Utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ applied to the Value in each state $\omega_i, i = 1, \ldots n$, at $t = 1$
- An example: $U(x) = \frac{-e^{-ax}}{a}$ where $a \in \mathbb{R}$ is the degree of risk-aversion
- Let the real-world probabilities be given by $\mu : \Omega \rightarrow [0, 1]$
- Denote $V_D = (V_D^{(1)}, \ldots, V_D^{(n)})$ as the payoff of Derivative D at $t = 1$
- Let x be the candidate price for D, which means receiving cash of $-x$ (at $t = 0$) as compensation for taking position D
- We refer to the candidate hedge by Portfolio $\theta = (\theta_0, \theta_1, \ldots, \theta_m)$ as the holdings in the fundamental assets
- Our goal is to solve for the appropriate values of x and θ

Appendix 4: Maximization of Expected Utility

- Consider Utility of the combination of $D, -x, \theta$ in state i at $t = 1$:

$$U(V_D^{(i)} - x + \sum_{j=0}^{m} \theta_j \cdot (S_j^{(i)} - S_j^{(0)}))$$

- So, the Expected Utility $f(V_D, x, \theta)$ at $t = 1$ is given by:

$$f(V_D, x, \theta) = \sum_{i=1}^{n} \mu(\omega_i) \cdot U(V_D^{(i)} - x + \sum_{j=0}^{m} \theta_j \cdot (S_j^{(i)} - S_j^{(0)}))$$

- Find θ that maximizes $f(V_D, x, \theta)$ with balance constraint at $t = 0$

$$\max_{\theta} f(V_D, x, \theta) \text{ such that } x = -\sum_{j=0}^{m} \theta_j \cdot S_j^{(0)}$$

Appendix 4: Maximization of Expected Utility

- Re-write as unconstrained optimization (over $\theta' = (\theta_1, \ldots, \theta_m)$)

$$\max_{\theta'} g(V_D, x, \theta')$$

$$\text{where } g(V_D, x, \theta') = \sum_{i=1}^{n} \mu(\omega_i) \cdot U(V_D^{(i)} - x \cdot e^r + \sum_{j=1}^{m} \theta_j \cdot (S_j^{(i)} - e^r \cdot S_j^{(0)}))$$

- *Price* of D is defined as the "breakeven value" z such that:

$$\sup_{\theta'} g(V_D, z, \theta') = \sup_{\theta'} g(0, 0, \theta')$$

- Principle: Introducing a position of V_D together with a cash receipt of $-z$ keeps the Maximum Expected Utility unchanged
- $(\theta_1^*, \ldots, \theta_m^*)$ that achieves $\sup_{\theta'} g(V_D, z, \theta')$ and $\theta_0^* = -(z + \sum_{j=1}^m \theta_j^* \cdot S_j^{(0)})$ are the associated hedges
- Note that the Price of V_D will NOT be the negative of the Price of $-V_D$, hence these prices serve as bounds/bid-ask prices